Problem 1. 2-player game

Consider the a game between two players. Both players have three pure strategies available to them. Both players are maximizers. The pay-off matrix is given as:

$$P = \begin{bmatrix} (3,1) & (7,8) & (5,6) \\ (5,4) & (2,3) & (4,2) \\ (1,3) & (4,10) & (6,7) \end{bmatrix}.$$

In each entry, the first element denotes the utility of the row player and the second element denotes the utility of the column player. Answer the following questions.

a) Find all pure Nash equilibria of the game. Are these equilibria admissible?

b) Compute a mixed Nash equilibrium.

c) Does this game admit an exact potential function?

Problem 2. (Renewable energy game)

Consider the following game:

- Player 1 is the owner of a solar power plant.
- Player 2 is the grid operator, which has to ensure that the power grid works reliably, and that the total demand is always equal to the total generation.

With a small probability ϵ, there might be a sudden imbalance in the grid, such that generation is not enough to meet the demand. For this reason, Player 1 (the solar power plant) is required to maintain some *reserve*, that is, to inject a bit less of its available solar power production, so that it can *ramp up* (increase its power generation) in these rare cases of extra demand. Maintaining the reserve has a cost, denoted by c, for Player 1, as the generator is selling less power than the available solar power. In case of extra demand, Player 2 (the grid operator) can also decide, instead of asking Player 1 to ramp up its generation, to *shed* (disconnect) some low-priority loads to restore power balance, at a cost s for Player 2. In this case, the reserve of Player 1 is not used. If, in the rare case of extra demand, Player 2 decides not to shed (and instead it asks Player 1 to ramp up generation), but Player 1 is not maintaining the reserve, then we have a blackout, that costs Player 2 a large sum b, greater than s. To summarize,

- Player 1 has two possible actions:

(R) *To reserve*, i.e. to sell less power than the available power and therefore be able to ramp up if requested by the grid operator.

(NR) *Not to reserve*, i.e. to sell all the available power, and not be able to ramp up if requested by the grid operator.

- Player 2 has two possible actions:

(S) *To shed*, i.e. in case of extra demand, to disconnect low priority loads instead of using the reserve.

(NS) *Not to shed*, i.e. in case of extra demand, to keep all loads connected and ask Player 1 to ramp up.

Player 1 does not know which action Player 2 will play in case of the rare event, and Player 2 cannot tell whether Player 1 is maintaining the reserve before deciding its action (to shed or not).

a) Express the game in bi-matrix form.

HINT: *Each outcome in the matrices represents the expected outcome given that the extra demand will only happen with probability ϵ.For example, if Player 1 plays (R) and Player 2 plays (S), the outcome of the game will be $J_1 = c$ for Player 1 and $J_2 = \epsilon s$ for Player 2.*

b) Are there dominated strategies? What is/are the pure Nash equilibrium/a of this game?

c) Consider now the case in which a *fine* f is introduced. Player 1 is fined only if the extra demand event happens, **and** Player 2 decides not to shed loads, **and** Player 1 didn't maintain the reserve. Only in this case, Player 1 has to pay the positive fine f to Player 2. Complete the payoff matrix corresponding to the game with the proposed fine.

$$\begin{array}{cc} & \begin{array}{cc} S & NS \end{array} \\ \begin{array}{c} R \\ NR \end{array} & \left[\begin{array}{cc} (c, \epsilon s) & (c, \cdot) \\ (\cdot, \epsilon s) & (\cdot, \cdot) \end{array}\right] \end{array}$$

d) i) What range of values of f guarantees that you don't have dominated strategies?

ii) Is the resulting game an ordinal potential game?

e) For the range of values of f computed in i) above, perform the following tasks.

i) Find all the pure Nash equilibria of this game (and their equilibrium values).

ii) Find all mixed Nash equilibria of this game (and their equilibrium values).

iii) What are the pure security policies for the two Players? What is the outcome, if both players choose to play their respective security policies?

f) For what values of f can we be sure that Player 1 will always maintain the reserves?

g) What is the Nash equilibrium and the resulting outcome in this case?

Let us define the Welfare function $W(y, z)$ as

$$W(y, z) = J_1(y, z) + J_2(y, z)$$

where y and z denote the strategies played by Player 1 and Player 2, respectively and $J_1(y, z)$ is the payoff for Player 1 and $J_2(y, z)$ is the payoff for Player 2. Let us recall the definition of Price of Anarchy as

$$PoA = \frac{\max_{(y^{NE}, z^{NE})} W(y, z)}{\min_{(y,z)} W(y, z)},$$

where the max operator is done with respect to all pairs of strategies (y, z) that are Nash Equilibria, while the min operator is done with respect to all possible strategies.

Let $c = 1$, $s = 200$, $b = 1000$, $\epsilon = 0.01$.

h) i) Assume Player 1 plays (NR) and Player 2 plays (NS), and therefore Player 1 has to pay the fine (with probability ϵ). How does the Welfare depend on the fine f?

ii) What is the Price of Anarchy with no fine ($f = 0$)?

iii) Prove that the Price of Anarchy is minimized when the value f of the fine satisfies the conditions derived in (f).

Problem 3. (ConsensusGame)

We are going to play a game motivated by the following problem. Let (V, E) denote a finite undirected graph with vertices $v_i \in V$, $i \in \{1, \dots, n\}$ and edges $E \subseteq \{(v_i, v_j) : v_i, v_j \in V\}$. Each vertex represents a person (player) and the edge (v_i, v_j) exists if i and j are friends (think for example of a social network). Let $N_i = \{(v_i, v_j) : v_j \in V\} \subset E$ denote the neighbors of player i, namely, those players who are linked to i. Player i has two choices: $x_i \in \{0, 1\}$, (think for example, deciding between going to two different cafes). The cost for player i is $J_i(x_i, x_{-i}) = \sum_{j \in \mathcal{N}_i} |x_i - x_j|$. By minimizing this cost, player i is trying to attend the same event as her neighbors. Consider the following *dynamics*. Each player starts by picking her favorite activity. Then, at each time step, one player updates her strategy by computing $\min_{x_i} J_i(x_i, x_{-i})$.

a) Show that consensus (in this case, agreeing on attending the same event) is a pure strategy Nash equilibrium.

b) Show that there can be other pure strategy Nash equilibria - you may provide an example.

c) Derive an exact potential function for the game.

d) Consider a graph with $n = 10$ vertices and randomly generate the edges. Randomly initialize an action $x_i \in \{0, 1\}$ corresponding to each vertex. Implement the best-reply dynamics in Matlab (or in other program you prefer). First, discuss what you expect to see from your simulation. Next, report your observations (attach a figure showing the total number of agreements at each iteration as well as the trajectory corresponding to 3 randomly selected vertices. Discuss whether the simulations are consistent with your expectations.

e) Repeat the same experiment, but this time, let players update their strategies simultaneously rather than iteratively. How are the results different from the previously discussed sequential case?

Problem 4. The metro ticket controller game

TL can send a ticket controller on your metro **one single time** in the month of April.

Every day, you decide whether to buy a ticket or not, without knowing whether the ticket controller will get on the metro.

A metro ticket costs 5 CHF, and the fine is 30 CHF.

a) Formulate game in extensive form (for a “month” of 2 days)

b) Compute the number of pure strategies of both players and the dimension of the game matrix.

c) Use backward induction for the last stage to reduce the game to a single stage game.

d) Compute the Nash equilibrium strategy for the single stage game above.